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Dissipative wave-mean interactions and the transport of vorticity or potential vorticity

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Wave-mean interactions of the classical type, in which the effect of the waves on the mean motion depends on wave breaking or other types of wave dissipation, are to be sharply distinguished from other types of wave-mean interaction that have no such dependence on dissipation. Important cases arise both for unstratified (homentropic) flow and for stably stratified flow under gravity. A very general way of characterizing what is meant by the classical, dissipative type of wave-induced mean motion is to say that the wave-induced mean motions are balanced motions, in a sense to be discussed, and that the effective mean force corresponds to the wave-induced vorticity or potential vorticity transport that results from wave dissipation. For a stratified fluid, 'potential vorticity' is to be understood in the sense of Rossby and Ertel. 'Balanced' is to be understood in whatever sense is needed to imply the invertibility of the vorticity or potential vorticity field to give the other fields describing the mean motion. At first sight this appears to require that an appropriate Mach, Froude and/or Rossby number for the mean motion should be much smaller than unity, but the fundamental, and in practice less stringent, principal requirement appears to be that the spontaneous emission, or aerodynamic generation, of sound, gravity and/or inertio-gravity waves by the mean flow should be weak.

Three basic examples of dissipative wave-induced mean flow generation are presented and discussed. The first is the transport of vorticity by dissipating sound waves, which gives rise to classical acoustic streaming of the quartz-wind type. The transport or flux of vorticity can always be taken to be an exactly antisymmetric tensor; and in the case of a plane sound wave this tensor fluctuates about a mean value equal to $-\epsilon_{ijk}\dot{q}_k$, where $\dot{q}_k$ is the kth component of $\dot{\boldsymbol{q}}$, the rate of dissipation of the pseudomomentum or quasimomentum $\boldsymbol{q} \approx E\boldsymbol{k}/\bar{\rho}\omega$ per unit mass. Here $\bar{\rho}$ and E are the mean mass and wave-energy densities, ω the intrinsic frequency, and $\boldsymbol{k}$ the wavenumber. This is a succinct way of making evident why it is only the contribution $\dot{\boldsymbol{q}}$ to the radiation stress convergence per unit mass that is significant for the generation of mean streaming. The second example is the transport of Rossby–Ertel potential vorticity (PV) by internal gravity waves that are either dissipating laminarly, or 'breaking' to produce inhomogeneous three-dimensional turbulence. This PV transport gives rise to mean streaming in much the same way as the vorticity transport in the acoustic example. The transport or flux of PV can always be taken to be directed exactly along the isentropic surfaces θ = constant of the stable stratification, where θ is potential temperature or potential density as appropriate; and in the case of a plane internal gravity wave the wave-induced PV transport fluctuates about a mean value $\boldsymbol{G} \times \dot{\boldsymbol{q}}$, where $\boldsymbol{G}$ is the basic gradient of θ associated with the stable stratification. This is a succinct way of making evident why it is only the projection of $\dot{\boldsymbol{q}}$ onto the basic stratification surfaces that is

significant. In both the acoustic and the internal-gravity examples the transport is non-advective, and often upgradient. The third example is the corresponding problem for Rossby waves, in which the typical effect of wave dissipation is a downgradient PV transport. This is brought about in an entirely different way, namely through advection of PV anomalies by the fluctuating velocity field of the wave motion, whether the dissipation be laminar or by breaking.

Processes of the sort idealized in the second and third examples are ubiquitous in the Earth's atmosphere and, for instance, largely control the strength of the global-scale middle atmospheric circulation and hence, for instance, the e-folding residence times ($\sim 10^2$ y) of man-made chlorofluorocarbons in the lower atmosphere.

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1. Introduction

Readers of this journal hardly need reminding of how wide a class of fluid phenomena can be "understood in terms of ... vorticity movement" (Batchelor 1967). This class can be widened still further, to include an important range of phenomena in large, stably stratified bodies of fluid like the Earth's atmosphere and oceans, and stellar interiors, if we include with 'vorticity' the *potential vorticity* in the sense of Rossby and Ertel, hereafter 'PV'. The phenomena in question include almost all large-scale motions of meteorological interest; and an understanding of them is fundamental to such apparently diverse topics as the improvement and quality control of weather forecasting (e.g. Hoskins *et al.* 1985; Hoskins & Berrisford 1988; McIntyre 1988), and the interplay between fluid dynamics and chemistry involved in the maintenance or destruction of the ozone layer (e.g. Brewer & Wilson 1968; WMO 1985, 1989).

An important subclass of these phenomena can usefully be thought of in terms of the interaction between waves and mean flows, an idea that has a long and distinguished history going back to nineteenth century work on acoustic streaming (Rayleigh 1896, and references therein). Rayleigh pointed out that the generation of certain mean flows by acoustic oscillations depends crucially on dissipative processes like viscosity, because in order to generate these flows Kelvin's circulation theorem must be violated. Wave-induced mean effects of a fundamentally similar kind are now believed to be central to understanding the global-scale atmospheric general circulation and the transport of ozone and pollutants (e.g. WMO 1985; Andrews, Holton & Leovy 1987). More complicated wave motions are involved, their restoring mechanisms depending on gravitational and Coriolis forces in a variety of ways. But the important mean effects so far identified have always turned out, on careful analysis, to be associated with wave dissipation of one kind or another. This causes Kelvin's circulation theorem either to be violated (as in Rayleigh's case) or to be circumvented by the irreversible material contour deformations characteristic of various 'wave breaking' processes, or both.†

This essay is an attempt to show how descriptions in terms of vorticity or PV provide a key to understanding, and more precisely characterizing, the general

† It is convenient for present purposes to widen the sense of the word 'dissipation' to include all cases of wave breaking, despite the fact that the latter concept may be meaningful even in the inviscid limit, at least for some models of Rossby waves. This is because of the known regularity properties of two-dimensional solutions of the Euler equations (e.g. Childress *et al.* 1989 and references therein). Further discussion of the concept of wave breaking, in the general sense relevant to wave-mean interaction theory, may be found in two papers by McIntyre & Palmer (1984, 1985).

nature of these mean effects of dissipating waves. This is no more than might have been anticipated from Lord Rayleigh's remark, and the essential ideas must have occurred to other investigators. But recent developments in the theory of PV inversion have, we think, enabled us to sharpen the ideas to a considerable extent and to reveal more clearly the full scope of their applicability and the nature of their ultimate limitations. The discussion tries to complement, rather than compete with, existing theories of wave-mean interaction. A guiding principle has been that of Batchelor (1953): "The manner of presentation ... has been chosen, not with an eye to the needs of mathematicians or physicists or any other class of people, but according to what is best suited, in my opinion, to the task of *understanding the phenomenon.* Where mathematical analysis contributes to that end, I have used it as fully as I have been able, and equally I have not hesitated to talk in descriptive physical terms where mathematics seems to hinder the understanding."

The phenomenon, or class of phenomena, that we are interested in understanding here is to be sharply distinguished from certain other, essentially non-dissipative, types of wave-mean interaction. We begin with brief descriptions of some known examples that illustrate the need to make this distinction.

2. Dissipative versus non-dissipative interactions

Classical examples of the dissipative type include the longshore currents due to ocean breakers (e.g. Longuet-Higgins 1970*a*, *b*; 1972), and their 'kitchen sink' counterparts, such as the cases depicted in figure 1. The case sketched in figure 1 (*a*) makes a reliable lecture demonstration, using a glass oven dish on an overhead projector; it closely resembles another well-known example, the 'quartz wind' or 'sonic wind' generated by a beam of dissipating ultrasound (e.g. Lighthill 1978*a*, *b*). All these examples illustrate the well-known rule of thumb that dissipating waves tend to have the same effect as a mean force in their direction of propagation.

A celebrated, and fascinating, example in the atmosphere is the so-called 'quasi-biennial oscillation', which has been well documented observationally since the early 1950s, and which involves the reversal, every thirteen months or so, of the mean easterly or westerly winds in the equatorial lower stratosphere throughout a belt encircling the globe. The evolution consistently shows a characteristic space-time pattern with the mean wind reversals taking place earlier at higher altitudes. Its cause is believed on good evidence (e.g. Wallace & Holton 1968; Lindzen & Holton 1968; Holton & Lindzen 1972; Dunkerton 1983; Plumb 1984) to be an interaction between the mean flow and certain kinds of upward-propagating, and possibly equatorward-propagating, gravitational-Coriolis waves originating in the troposphere. An idealized form of the same phenomenon – a periodically reversing mean flow driven entirely by a steady input of waves, and exhibiting qualitatively the same space-time pattern – has been demonstrated in the laboratory using viscously dissipating, upward-propagating internal gravity waves in a salt-stratified fluid in a large annular container (Plumb & McEwan 1978). The dissipating waves induce an azimuthal mean flow, which in turn strongly retracts the waves; and the reversals of the mean flow arise as a feedback oscillation in which both halves of the interaction between waves and mean flow play an essential role.

These and many other examples of the mean flows induced by dissipating waves have a general character that is easy to recognize but not, at first sight, so easy to define precisely. Part of the difficulty lies in what happens when one tries to make the ideas of 'mean flow' and 'mean force' precise yet general. The use of averaging

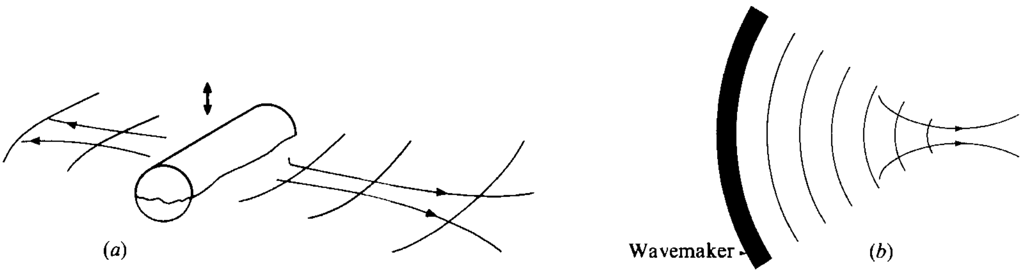

FIGURE 1. Simple lecture or laboratory demonstrations illustrating the typical nature and robustness of classical mean-flow generation by dissipating waves. (*a*) An example using water waves that requires no special apparatus. The cylinder, or any other anisotropic wavemaker, is oscillated fairly rapidly ($\gtrsim$ 5 Hz) so as to radiate short waves more strongly in some directions than in others. The resulting mean streaming can be made visible (and wave dissipation enhanced) by sprinkling a little powder such as ordinary household flour on to the surface of the water. Making the cylinder oscillate vertically demonstrates that the observed mean flow is predominantly wave-induced, and not boundary-layer streaming from the surface of the wavemaker, which has the opposite sense. (*b*) By using a longer curved wavemaker in a larger tank, one can focus the waves on a spot well away from the wavemaker and thus induce an easily observable mean flow concentrated near that location. Carefully stopping the wavemaker and observing the persistence of the mean flow, in either case, demonstrates that the mean flow is not merely a 'Stokes drift'. Relevant theoretical discussions are those of Longuet-Higgins (1953) and Craik (1982*a*), among others.

operators is almost inevitable at some stage; but, for the purpose of "understanding the phenomenon", averaging is very much a two-edged sword. On the one hand, its systematic use can simplify the analysis very significantly, especially if an appropriate Lagrangian average is used, so that the aspects connected with the circulation theorem, including the so-called 'non-acceleration theorem', can manifest themselves (e.g. Bretherton 1969, 1971; Dewar 1970; Andrews & McIntyre 1978; McIntyre 1980*a*, *b*). On the other hand, the complete set of averaged equations and boundary conditions generally contains a variety of mean forces and mean flow phenomena that have nothing to do with wave dissipation. Among the latter are mean flow contributions of the Stokes-drift type, along with other transient, reversible mean-flow changes that depend on the continued presence of the waves (and on the choice of averaging operator) and that return to zero if the waves propagate out of the region of interest. Such mean effects are uniformly bounded, as time goes on, to be $O(a^2)$ in the wave amplitude a. In addition, there are non-dissipative wave-induced mean motions that are not so bounded. One example is the Craik–Leibovich instability, which is in a category of its own and will be discussed briefly in §9. Other non-dissipative cases not uniformly bounded by $O(a^2)$ include various 'long wave, short wave' and 'low frequency, high frequency' interactions, in which the dynamics of the mean state, as defined by the chosen averaging operator, is itself wavelike, allowing resonant buildup of oscillatory mean motions beyond $O(a^2)$ (e.g. Westervelt 1963, 1977; Mahony & Smith 1972; Grimshaw 1977, 1979; Dysthe & Das 1981, and references therein).

A simple but striking example of the latter kind occurs when acoustic oscillations are driven in a closed fluid system that also has slower modes of oscillation, such as surface or internal gravity waves (Mahony & Smith 1972). For a demonstration with minimal apparatus one may use a commercial acoustic shaker and a large glass laboratory beaker holding four or more litres of water; more quantitative versions of the experiment have been done in rigidly enclosed air–water cavities (Franklin, Price & Williams 1973). When such a system is vigorously driven near one of its lowest resonant acoustic frequencies, and dissipation is small enough, a standing gravity oscillation may build up visibly and spontaneously. The gravity oscillation has frequency far lower than the acoustic driving frequency. In the beaker demonstration one often observes the gravest axisymmetric, zero-order Bessel-function mode. The sound field sees, in effect, a mean state varying slowly in time, causing the sound field itself to vary, with a certain phase lag, as conditions approach and recede from resonance. The slowly varying mean forces due to the sound waves are able to do work on the gravity oscillations, and cause them to grow exponentially.† As with the other cases of this type, the mechanism has no essential dependence on dissipation; indeed the demonstration will not work if there is too much dissipation.

More extreme cases such as the 'acoustic fountain' (see for instance Hertz & Mende 1939; Bergmann 1954) can arise when the mean force due to the radiation stress divergence in a vertical beam of ultrasound encountering density or other inhomogeneities is so strong that it directly overcomes the gravitational restoring force associated with a stable density stratification. The enhancement of stellar winds by the action of acoustic radiation stresses (e.g. Pijpers & Hearn 1989, and references therein) is a somewhat similar case. Such cases are more straightforwardly a matter of $O(a^2)$ quantities themselves not being 'small' for the purpose at hand, where a is the sound wave amplitude, but they may also be counted as examples of significant wave-driven or wave-modified mean flows that have no essential dependence on wave dissipation.

How, then, might one characterize the recognizably different class of wave-driven mean motions like those in figure 1, of which classical acoustic streaming is the prototype? One would like to have a general conceptual framework

(*a*) that specifies what it is that distinguishes these classical phenomena from the non-dissipative phenomena just mentioned;
(*b*) that includes, in principle, complicated three-dimensional flow geometries, not just those to which a simple spatial average and hence a 'non-acceleration theorem' applies; and
(*c*) that applies equally to classical problems and to the more complicated problems of stratified and/or rotating flow that are important in the Earth's atmosphere and oceans.

One is presumably close to such a characterization when one says, paraphrasing Lord Rayleigh, that the phenomena in question are 'circulation changing'. But from a strictly logical viewpoint that statement is incomplete, since it could be said to include some of the non-dissipative phenomena just mentioned, such as those in which the mean motion has the character of a gravity wave. This is because not enough is said about the way in which the circulation is relevant. The next two sections present what seems to us to be an illuminating, possibly complete, and certainly very general way of stating the desired characterization. It proves convenient to use vorticity or PV in place of circulation. In §§5–8 we discuss a few of the most basic examples explicitly, touching also on some of the points that are relevant to understanding the dynamics and chemistry of the atmosphere.

† The most favourable condition for this to happen (with a phase lag less than half a gravity-wave cycle) is when the driving frequency is just *above* an acoustic resonance. This can immediately be understood, without detailed calculation, from the connection between radiation stress and the Boltzmann–Ehrenfest adiabatic invariance theorem (Brillouin 1925).

3. Conceptual framework: transport, balance and invertibility

There are three key ideas. The first is that a typical effect of wave dissipation is to cause an irreversible transport of vorticity or PV that would not otherwise have taken place. The irreversibility can arise in several ways, as will become clear from the examples.

The second idea is that the mean flows concerned are approximately 'balanced' flows, in a sense illustrated by the slow evolution of solutions to stiff differential equations (e.g. Press *et al.* 1986). For instance the mean flows appearing in classical acoustic streaming problems are low Mach number flows, in the sense that the mean flow dynamics itself involves negligible acoustic oscillations. Again, the typical mean flows driven by dissipating internal gravity waves are quasi-horizontal, low Froude number flows involving negligible buoyancy oscillations (e.g. Bretherton 1969). If one were to draw the partial analogy with a mechanical system of masses and springs of varying stiffness, one would say that the mean flow dynamics is like the slow evolution of the mechanical system, with negligible excitation of free oscillations involving the stiffer springs. It is the idea of balance that distinguishes classical mean streaming and its analogues, and the effective wave-induced mean forces associated therewith, from the non-dissipative wave-driven mean motions and associated mean forces referred to near the end of §2. The latter mean motions can be described as wave-driven mean motions, but certainly not as wave-driven balanced mean motions. Their essence is, on the contrary, the *unbalancing* of the mean state by the direct excitation of wavelike mean motions. The acoustic fountain is an extreme case of this; the mean flow can be thought of as resembling a continually forced, breaking gravity wave, driven by the radiation stress convergence associated with the propagation of a beam of sound waves through a density inhomogeneity.

The third idea is prompted by the second, and connects it to the first. What is the most useful and general way of defining the notion of balanced flow? Arguably, the answer is simply to say that balanced flows are just those fluid flows that are controlled by vorticity or PV evolution. In other words, balanced flows are those fluid motions to which an 'invertibility principle' for vorticity or PV applies, in the sense that the vorticity or PV field can be inverted to yield the velocity field and any other relevant dynamical information. The Biot-Savart or inverse Laplacian type of vorticity inversion integral, for unstratified vortical flows in the zero Mach number limit, is merely the most familiar special case, corresponding to complete rigidity of the stiffer springs in the mechanical analogy. Other cases are discussed in the review by Hoskins *et al.* (1985) and in a forthcoming paper by the present authors (1990, hereafter MN). In the geophysically important case of the PV in a stably stratified, rapidly rotating fluid the basic insights go back to the pioneering work of Charney (1948) and Kleinschmidt (1950*a*, *b*, 1951), which was motivated by some of the meteorological problems recalled in §1.

The general notion of vorticity or PV invertibility is by no means a trivial one, if only because the most accurate inversion operators are nonlinear at finite Froude, Mach or Rossby numbers, and also because inversion is then, almost certainly, an inherently approximate process, as discussed briefly in Hoskins *et al.* and more extensively in MN. In practice, however, the approximations involved can be astonishingly good, in comparison with what one might guess from a *prima facie* consideration of Froude or Mach number values. It is this that gives rise to the suggestion of usefulness combined with great generality. Figure 2 shows an example, obtained in the course of our recent work on atmospheric modelling. The system is a shallow, hydrostatic layer of unstratified fluid with a free upper surface, on a hemisphere viewed in polar stereographic projection in a rotating frame of reference rotating with angular velocity $\Omega = 2\pi/(1 \text{ day})$. It was originally studied as a simplified model atmosphere, but can also be thought of, in the usual way, as a two-dimensional acoustic system, with sound waves in place of gravity waves, in a hypothetical 'perfect gas' with ratio of specific heats $\gamma = 2$, and with density ρ replacing the layer depth h. The system has an invariant, Q, that is materially conserved in the absence of viscosity:

$$Q = \zeta^a/h \quad \text{or} \quad \zeta^a/\rho, \quad \mathrm{D}Q/\mathrm{D}t = 0, \tag{3.1}$$

where ζ^a is the radial component of absolute vorticity and $\mathrm{D}/\mathrm{D}t$ the two-dimensional material derivative. For want of any other name Q will be referred to indiscriminately as the PV both in the free-surface and in the compressible interpretation.

Figure 2(*a*) shows the velocity field (arrows) and the h or ρ field (contours) taken from a high-resolution numerical simulation of nearly inviscid evolution of this model fluid system. The local Froude or Mach number M, defined as the ratio of the local flow speed $|\boldsymbol{u}|$ to the gravity or sound wave speed, takes values up to about 0.5. The local Rossby number R, defined as $|\boldsymbol{u}|/2L\Omega \sin\phi$, where ϕ is latitude and L is a length scale of the flow, is of course infinite at the equator. It reaches a local maximum of the same order as M, about 0.5, in the region of strong subtropical winds. The flow was set up by means of an artificial forcing, in a manner that need not concern us here except to say that care was taken to apply the forcing smoothly, so that minimal gravity or acoustic wave activity was introduced as a direct result. Figure 2(*b*) shows the two-dimensional divergence field. Figure 2(*c*, *d*) shows the associated distribution of Q, both in conventional contouring and in grayscale for better visibility. The smallest features are well resolved numerically; a pseudo-spectral method is used, isotropically representing the fields using total spherical harmonic wavenumbers up to 106, corresponding to a mesh size of the order of a degree of latitude.

Figure 2(*e*, *f*) shows the results of a nonlinear inversion. The velocity, density and divergence fields were reconstructed, from the Q field alone, using an accurate PV inversion algorithm described in MN (*q.v.* for further detail). Comparison with figure 2(*a*, *b*) shows that the invertibility principle applies with remarkable accuracy to this particular flow. Despite the not very small values of the local Froude, Mach and Rossby numbers, and the large departure from incompressibility (the density can be seen from figure 2(*a*, *e*) to vary by fractional departures 0.4 or more from its area mean), almost all the detail is recovered by the inversion, even in the divergence field. It follows that the flow is accurately balanced, in the sense proposed above, even though the fact that 0.4 is hardly small compared with unity means that a simple incompressible or inverse Laplacian inversion applied to Q would be grossly inaccurate. (An explicit computation, not shown, gives maximum velocities typically wrong by a factor ~ 2.)

This particular example supplements a more extensive set of examples given in MN. The impression given by figure 2 is typical of all the other examples, which include cases with local Froude or Mach numbers up to 0.7 albeit not simulated at such high spatial resolution. The discussion in MN also describes some tests of *cumulative* accuracy, in which the exact evolution over several characteristic eddy times was compared with the evolution of a corresponding 'balanced model' defined by alternately inverting, and time-stepping (3.1). Even with maximum local Froude, Mach and Rossby numbers in excess of 0.7, the evolution was reproduced

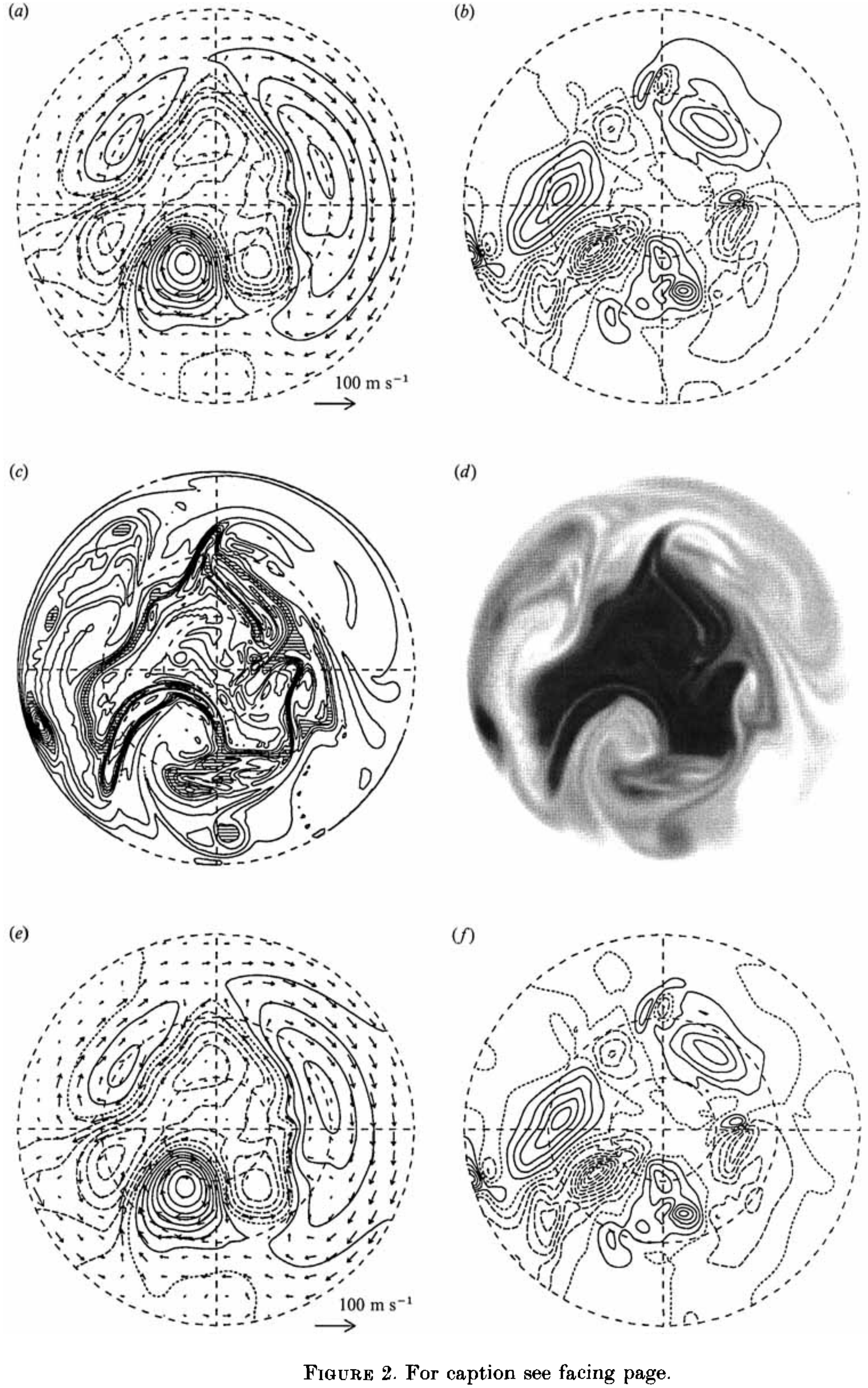

FIGURE 2. For caption see facing page.

with an accuracy not far short of that in figure 2. The fact that the balance and inversion concepts seem so remarkably accurate is presumably connected with another, better known fact, namely the equally remarkable weakness of spontaneous emission of acoustic or gravity waves predicted by the Lighthill theory of aerodynamic sound generation (e.g. Crighton 1975, 1981).

Our results to date, then, in combination with the Lighthill theory, strongly encourage the belief that the concept of balanced flow, as defined above, is far more widely applicable than one might imagine from the standard approximate inversion theories that restrict attention to Mach, Froude and/or Rossby numbers much less than unity. This belief has yet to be fully tested either in the case of three-dimensional homentropic (unstratified) vorticity inversion at substantial Mach numbers, or in the case of the multi-layer or continuously stratified fluid systems that are of interest in connection with applications to the atmosphere and oceans. Limitations on applicability will be encountered at some stage, of course; for instance restrictions on vertical structure are to be expected in the continuously stratified case, since if arbitrarily fine vertical scales were to be allowed then internal Froude numbers would tend to become large. In practice such structures might tend to be self-limiting inasmuch as large internal Froude numbers usually go with small Richardson numbers, so that there will tend to come a stage at which Kelvin–Helmholtz and other shear instabilities set in.†

It should be remembered, also, that the other types of wave-induced mean forces illustrated in §2 will perturb the balance condition and therefore the inversion operator. This is related to the distinction between 'non-interaction' and 'non-acceleration' explained in Andrews & McIntyre (1978, equation (5.9) ff.). Such effects will usually degrade the accuracy of the balance and invertibility concepts by a uniformly bounded $O(a^2)$ amount when waves are present. What this would mean quantitatively would have to be checked in particular problems. Important

† Most of the available results on PV inversion in stratified fluids do not address any of these questions, since they are restricted to rapidly rotating cases where both the Froude and Rossby numbers are small enough to justify approximation schemes based on quasi-geostrophic theory and its refinements. This is reviewed in Hoskins *et al.* (1985); and see also, for example, Mattocks & Bleck (1986). A recent paper by Staquet & Riley (1989) breaks new ground by treating the non-rotating case, but still only for small Froude number. As far as we know, no excursion beyond these parameter regimes has yet been attempted.

FIGURE 2. Demonstration of balance and invertibility in a compressible flow with substantial density variations, from a high-resolution numerical experiment on flow on a hemisphere. This was motivated as an atmospheric model but is equally well interpretable as a compressible two-dimensional flow in a hemispherical shell. The system is a shallow water free-surface model with area-mean depth 2 km and corresponding gravity wave speed 140 m s^{-1}, or equivalently a fictitious 'perfect gas' with ratio of specific heats $\gamma = 2$ and sound speed 140 m s^{-1} at mean density. Solid contours show positive values, long dashed contours negative, and dotted contours zero. The projection is polar stereographic; the radius of the hemisphere is 6371 km. (*a*): Arrows show the velocity field on the scale indicated; contours show departures of density or layer depth from the area mean value. The contour interval is one twentieth of the mean; in the two-dimensional compressible system it can also be regarded as the anomaly in the square root of the pressure. (*b*) Divergence field contoured at intervals of 0.6×10^{-6} s^{-1}. (*c*, *d*) The quantity Q defined in equation (3.1). (*c*) is contoured at interval 1×10^{-8} m^{-1} s^{-1} in units appropriate to the first formula in (3.1). The shading in the contour plot highlights values lying between 4 and 6 of these units. The greyscale representation of the same information in (*d*) is monotonic from light to dark, from zero at the equator to a maximum value of 1×10^{-7} m^{-1} s^{-1} near the pole. (*e*, *f*) As (*a*, *b*), but reconstructed from Q alone using an accurate nonlinear PV inversion algorithm.

cumulative effects seem unlikely, except when the timing of the mean forces is such as to drive a departure from balance resonantly, as in the beaker experiment described in §2. As already suggested, such phenomena, as well as the intrinsic limitations of the balance and invertibility concepts at substantial mean-flow Mach or Froude numbers, are part of what sets the limitations on how far we can generalize the notion of classical mean streaming. A further possible effect on inversion operators for stably stratified fluid systems is wave-induced diabatic heating, which can change the reference stratification used in the inversion.

It remains to define the notion of vorticity or PV transport in a suitably general way. This is done in the next section, after which we proceed to the explicit examples.

4. The transport of vorticity or PV and the effective mean force

The exact, general conservation properties of vorticity and PV are presumably familiar to many investigators, although as far as we know they have not been systematically discussed together, in full generality, except in a recent pair of papers by Haynes & McIntyre (1987, 1990, hereafter HM; *q.v.* for history). To recapitulate briefly, two basic kinematical properties of the vorticity vector are first that, away from boundaries, vorticity is always exactly conserved, and second that the net flux or transport of vorticity can always be taken to be directed exactly at right angles to the vorticity itself. This is true for an arbitrary equation of motion

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{X}, \tag{4.1}$$

since taking the curl shows at once that the ith component ζ_i of the vorticity vector $\boldsymbol{\zeta} = \boldsymbol{\nabla} \times \boldsymbol{u}$ always satisfies the conservation relation

$$\frac{\partial}{\partial t}\zeta_i + \frac{\partial}{\partial x_j} Z_{ij} = 0, \tag{4.2}$$

where the components Z_{ij} of the vorticity flux or transport tensor $\boldsymbol{Z}$ can be expressed as

$$Z_{ij} = \epsilon_{ijk} X_k; \tag{4.3}$$

$\boldsymbol{Z}$ is an antisymmetric tensor, verifying what was just said about the direction of the flux or transport. If the frame of reference is rotating with constant angular velocity $\boldsymbol{\Omega}$ then the same conservation equation is satisfied also by the ith component ζ_i^{a} of the absolute vorticity $\boldsymbol{\zeta}^{a} = \boldsymbol{\zeta} + 2\boldsymbol{\Omega}$. For motion under a conservative gravitational–rotational potential Φ and an arbitrary body force $\boldsymbol{F}$ per unit mass, we have explicitly, in (4.1),

$$\boldsymbol{X} = \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} + \alpha\boldsymbol{\nabla}p + \boldsymbol{\nabla}\Phi - \boldsymbol{F}, \tag{4.4}$$

where $\alpha = 1/\rho$, the specific volume, and p is the pressure. Then (4.3) expands to

$$Z_{ij} = -u_i\zeta_j + u_j\zeta_i - 2u_i\Omega_j + 2u_j\Omega_i + \epsilon_{ijk}\left(\alpha\frac{\partial p}{\partial x_k} - F_k\right), \tag{4.5a}$$

after ignoring identically non-divergent contributions like $\epsilon_{ijk}\,\partial\Phi/\partial x_k$. Equivalently,

$$Z_{ij} = -u_i\zeta_j^{a} + u_j\zeta_i^{a} + \epsilon_{ijk}\left(\alpha\frac{\partial p}{\partial x_k} - F_k\right). \tag{4.5b}$$

As discussed in HM, the PV inherits a corresponding pair of exact, basic properties. These are, again, exact conservation, and a restriction on the directions in which net transport can take place, to be referred to below as the 'impermeability theorem'. The 'PV' as defined in equation (3.1) satisfies the two-dimensional counterpart of (4.2) and needs no further discussion; the exact (Ertel) definition of PV for a continuously stratified fluid is

$$Q = \rho^{-1}\boldsymbol{\zeta}^{a}\cdot\nabla\theta, \tag{4.6}$$

where θ is potential temperature, or potential density or other appropriate measure of buoyancy such that

$$\alpha = \rho^{-1} = \text{func}\,(\theta, p). \tag{4.7}$$

The Q defined by (4.6) is the amount per unit mass of an exactly conserved quantity whose conservation relation can be written in the form

$$\frac{\partial}{\partial t}(\rho Q) + \nabla\cdot\boldsymbol{J} = 0; \tag{4.8}$$

the conservation form can be seen at once by (i) replacing ζ_i in (4.2) by $\zeta^{a}_i = \epsilon_{ijk}\,\partial u^{a}_k/\partial x_j$, where $\boldsymbol{u}^{a}$ is the absolute velocity $\boldsymbol{u}+\boldsymbol{\Omega}\times\boldsymbol{x}$, (ii) taking the scalar product of $\partial\theta/\partial x_i$ with (4.2), and (iii) noting the vanishing of the double inner products of the antisymmetric tensors Z_{ij} and $\epsilon_{ijk}\,u^{a}_k$ with the symmetric tensors $\partial^2\theta/\partial x_i\,\partial x_j$ and $\partial^3\theta/\partial x_i\,\partial x_j\,\partial t$. The impermeability theorem can be demonstrated in a similarly general way. It is noteworthy that (4.7) is not required; for further discussion see HM §5. Rather than repeating that discussion we note, instead, an explicit form that besides making the impermeability property evident will also be useful in other ways. Making use of (4.5*b*), again ignoring identically non-divergent contributions, and now using (4.7), one can obtain, after a little manipulation, a form of (4.8) in which

$$\boldsymbol{J} = \rho\boldsymbol{u}^{\theta\perp}Q + \rho\boldsymbol{u}^{\parallel}Q - H\boldsymbol{\zeta}^{a\parallel} - \boldsymbol{F}\times\nabla\theta; \tag{4.9}$$

see also Haynes & McIntyre (1990). Here $H = \mathrm{D}\theta/\mathrm{D}t$, a measure of heating rate per unit mass, or other buoyancy forcing, while

$$\boldsymbol{u}^{\parallel} = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\nabla\theta}{|\nabla\theta|^2}\nabla\theta, \quad \boldsymbol{\zeta}^{a\parallel} = \boldsymbol{\zeta}^{a} - \frac{\boldsymbol{\zeta}^{a}\cdot\nabla\theta}{|\nabla\theta|^2}\nabla\theta, \tag{4.10a, b}$$

and

$$\boldsymbol{u}^{\theta\perp} = -\frac{\partial\theta/\partial t}{|\nabla\theta|^2}\nabla\theta. \tag{4.10c}$$

It can be seen that the last three terms in (4.9) represent vectors that are exactly parallel to the local constant-θ surface. The first term, by contrast, is ρQ times the vector $\boldsymbol{u}^{\theta\perp}$, which is just the velocity of the θ-surface normal to itself. It follows that a point moving with velocity $\boldsymbol{J}/(\rho Q)$ always remains on exactly the same θ surface, even when the fluid is itself moving through that surface, as occurs when the heating $H \neq 0$. This says that the θ surfaces behave as if they were completely impermeable to the PV – or, more precisely, impermeable to the additive, extensive, exactly conserved quantity whose amount per unit mass is the PV – even when the heating H makes the same θ surfaces permeable to mass and chemical substances. This is the result we call the 'impermeability theorem' and is the property of PV transport that corresponds to the antisymmetry property of the vorticity transport. It was derived in several other ways in HM (and was apparently new at the time); the present version is the most convenient form here. We recognize of course the well known non-uniqueness of the flux or transport in a conservation relation; for instance one could add any identically non-divergent field to (4.5) or (4.9). But this would obscure a strikingly simple and useful fact about vorticity and PV transport.

The relevance of (4.5) to classical wave-induced homentropic mean flows can now be stated as follows. Let the mean or slowly varying part of the wave-induced contribution to the vorticity transport be denoted by $\overline{\boldsymbol{Z}^{\mathrm{w}}}$. We shall see an example in the next section of how this arises from quadratic correlations on the right of (4.5). To any such vorticity transport there always corresponds a mean force per unit mass

$$\overline{F}_{k\,\mathrm{eff}} = -\tfrac{1}{2}\epsilon_{ijk}\overline{Z^{\mathrm{w}}_{ij}} \tag{4.11}$$

that, if applied to the fluid in the absence of waves, would produce the same vorticity transport. It would therefore have the same effect as the waves on any balanced mean flow. We may therefore identify it as the relevant 'effective mean force'. Its effect on the vorticity transport can be verified to be equivalent to a contribution $\overline{\boldsymbol{Z}^{\mathrm{w}}}$ by substituting the expression (4.11) into the F_k term in (4.5). In so far as the response to this effective mean force remains balanced, it represents the sole effect of the waves on the mean flow. This neglects the uniformly bounded, $O(a^2)$ perturbations to the vorticity inversion operator discussed earlier. Note that ignorable, identically non-divergent contributions to $\mathbf{Z}$ produce ignorable, irrotational contributions to the effective force.

Similarly, the relevance of (4.9) to classical wave-induced stratified mean flows can be stated in terms of the mean or slowly varying part of the wave-induced contribution to the PV transport. Let this contribution be $\overline{\boldsymbol{J}^{\mathrm{w}}}$. We shall see an example in §6 of how this arises from quadratic correlations on the right of (4.9), with $\overline{\boldsymbol{J}^{\mathrm{w}}}$ directed along the mean or basic stratification surfaces. To any such PV transport there always corresponds a mean force per unit mass

$$\overline{\boldsymbol{F}}_{\mathrm{eff}} = \overline{\boldsymbol{J}^{\mathrm{w}}} \times \nabla\bar{\theta}/|\nabla\bar{\theta}|^2 = \overline{\boldsymbol{J}^{\mathrm{w}}} \times \boldsymbol{G}/|\boldsymbol{G}|^2, \text{ say,} \tag{4.12}$$

also directed along the stratification surfaces, that again would have the same effect as the waves upon the PV transport and therefore, again, the same effect on any balanced mean flow. We may therefore identify it as the relevant 'effective mean force' for stratified flow, in exactly the same way. Its effect on the PV transport can be verified by substituting the expression (4.12) into the $\boldsymbol{F}$ term in (4.9), with the basic gradient $\boldsymbol{G} = \nabla\bar{\theta}$ substituted for $\nabla\theta$.

It will have been noticed incidentally that, in talking heuristically about 'means', we have not yet tried to distinguish between Eulerian means, Lagrangian means, and basic flows in other senses that might be useful. This is partly because the correlations involved in (4.11) and (4.12) are robust, and insensitive to what kind of mean is taken†, and partly because we are interested in effects that are cumulatively much larger than Stokes corrections and other uniformly bounded $O(a^2)$ effects. The distinction does turn out to be critical, however, for our third example (Rossby waves), most notably in the case of thermal dissipation (§8). In order for the thermally dissipating Rossby-wave case to fit well into the present conceptual framework we shall find it best to employ averaging of a hybrid sort, along undular θ-surfaces but, most importantly, Eulerian in plan view, as described in §8.

The notion that a disturbance-induced mean force, or certain rotational contributions to it, can usefully be thought of as equivalent to an averaged flux or

† As long as the mean has the relevant additivity property, see below (§§5, 6).

transport of vorticity or PV, is not of course new. It has long been familiar in certain special cases, particularly studies of incompressible, two-dimensional turbulence, and goes back to Taylor (1915); see also, for example, Goldstein (1938, §83). The same idea has found extensive use in geophysical fluid dynamics within the framework of quasigeostrophic theory for a stably stratified, rapidly rotating fluid (e.g. Bretherton 1966, Dickinson 1969*a*, Green 1970, Rhines and Holland 1979, Edmon *et al.* 1980, Dunkerton *et al.* 1981), which theory has validity as a first approximation when the Froude and Rossby numbers are both small. The interest here is in trying to probe the ultimate limitations of the idea; and as we have just seen, the limitations are to be found not in the vorticity or PV transport properties themselves, which can of course be formulated exactly, as was done above, but only in the limitations on the concepts of balance and invertibility that allow the identification of the expressions (4.11) and (4.12) as 'effective mean forces'.

We now turn to the specific examples, confining ourselves in each case to the simplest relevant example that suffices to illustrate the principle involved.

5. Upgradient vorticity transport by dissipating sound waves

When reminded of the easily observable jets driven by beams of ultrasound, sometimes called the 'quartz wind' or 'sonic wind' (e.g. Lighthill 1978*a, b*) it is natural to ask "where does the vorticity come from?" The jets start more or less from rest when the sound source is turned on; and the vorticity clearly cannot all be introduced at a boundary. Figure 3 shows schematically a perspective view of some of the circular vortex lines in an axisymmetric quartz-wind jet directed along the x-axis. The creation of such a vorticity pattern from nothing evidently requires the vorticity to be transported up its own gradient.

Let us define p in (4.5) to be a function of α alone (α denoting $1/\rho$, as before); then by definition all the irreversible, dissipative effects, whether described as thermodynamical or quasiviscous, are incorporated into the force $\boldsymbol{F}$. The p term in (4.5), so defined, evidently produces only an identically non-divergent contribution to the transport $\boldsymbol{Z}$, where therefore does nothing to the vorticity distribution and can be ignored. Following Lighthill (1978*b*, §1.13), and confining attention at first to an idealized plane sound wave travelling in the positive x- or 1-direction, we may write $\boldsymbol{F}$ as a fluctuating force oriented in the same direction and equal to the one-dimensional divergence of a normal stress $-\kappa\,\partial\rho/\partial t$, where κ, the 'diffusivity of sound', may be taken as a constant for our purposes. Taking $\partial/\partial x$ of $-\kappa\,\partial\rho/\partial t$ and substituting the result into the F_k term in (4.5), with $k = 1$, we obtain for the wave-induced vorticity transport

$$Z_{ij}^{\mathrm{w}} = \kappa\alpha\frac{\partial^2\rho}{\partial x\,\partial t}\epsilon_{1ij}. \tag{5.1}$$

In a plane, sinusoidal, progressive sound wave, the fluctuations in α are in antiphase with those in ρ and therefore with those in $\partial^2\rho/\partial x\,\partial t$, since the phase speed is positive. Thus $\alpha\,\partial^2\rho/\partial x\,\partial t$ fluctuates about a negative mean. This is evidently a robust result, in the sense that it would be true on the basis of any relevant definition of 'mean', that is to say a definition that preserves the additivity of the vorticity transport across control surfaces subdivided into area elements. At all locations in the wave where $\partial^2\rho/\partial x\,\partial t$ is negative, α is high, and where $\partial^2\rho/\partial x\,\partial t$ is positive, α *is low; and* $\partial^2\rho/\partial x\,\partial t$ has zero mean for any wave, sinusoidal or not, that is periodic in x or in t or in both.

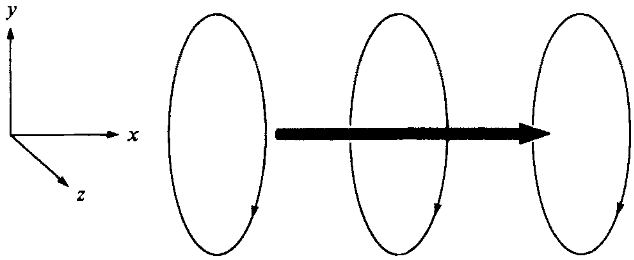

FIGURE 3. Perspective sketch showing three of the vortex lines in an axisymmetric 'quartz wind' jet. The central arrow indicates both the sense of the beam of ultrasound generating the jet, and also the sense of the jet itself.

Now consider the laboratory case of a confined beam of ultrasound. By what has just been said, the qualitative sense of the wave-induced vorticity transport will still be given by the expression (5.1), in any situation where a plane sound wave serves even as a rough first approximation. (In some cases of very short, small-amplitude waves and approximately parallel beams, the waves may be nearly plane in reality, and (5.1) a quantitatively useful approximation as well.) The negative sign of $\alpha\,\partial^2\rho/\partial x\,\partial t$ shows that Z^{w}_{ij} has the sign of $-\epsilon_{1ij}$, meaning that positive 3-vorticity (z component) is transported in the positive 2-direction or y-direction, and negative 2-vorticity (y component) in the positive 3-direction or z-direction. The transport converges in an annular region surrounding the centre of the beam. This creates vortex loops oriented in the sense of a right handed screw along the direction of propagation, as suggested in figure 3 – corresponding to an effective mean force (4.11), and hence a jet, in that same direction.

It is worth re-emphasizing how neatly the foregoing handles all the second order mean pressure effects that enter into, and complicate, a description in terms of radiation stresses. They are dealt with at a stroke by the remark already made, that any contribution to the vorticity transport Z_{ij} of the form

$$\epsilon_{ijk}\,\alpha\frac{\partial\,\mathrm{func}\,(\alpha)}{\partial x_k} \tag{5.2}$$

is identically non-divergent and is therefore a 'do-nothing' transport, which can be ignored.

It is not difficult to convince oneself that none of the other terms on the right of (4.5) can have nearly as large an effect, if only because the basic acoustic motion is irrotational to a first approximation, as well as nearly one-dimensional. It is only the F_k term that can create a vorticity pattern out of nothing. The order-of-magnitude verification is omitted for brevity.

Another way of appreciating the robustness of the effect under discussion is to imagine an experiment in a fluid with spatially variable dissipative properties, in which the waves propagate a long way without dissipating and then encounter an isolated region $\mathscr{R}$ with high local dissipation. (The analogue of this actually happens all the time on ocean beaches, and to some extent in the experiment of figure 1 (*b*).) Such thought experiments make it particularly clear that it is the correlation expressed by (5.1) ff., and not, for instance, any resultant $O(a^2)$ mean viscous force on the region $\mathscr{R}$, that is important. For instance the resultant force from viscous stresses on a surface enclosing the region can be made arbitrarily small (at least in the imagination) simply by making the viscosity on the enclosing surface arbitrarily small, while continuing to dissipate the waves within $\mathscr{R}$ and continuing to generate mean streaming via (5.1). One such thought experiment is to imagine the waves to be dissipated purely thermally.

To check that (5.1) does correspond to the standard formulae associated with the quartz-wind problem, we note that for a sinusoidal wave the effective force $-\kappa\alpha\,\partial^2\rho/\partial x\,\partial t$ per unit mass has the mean value

$$-\kappa k\omega\,\overline{\alpha'\rho'} = \kappa k^2 c(E/\bar{\rho}c^2) = \dot{E}/\bar{\rho}c = \dot{q}_1. \tag{5.3}$$

Here ω, k and c are the frequency, wavenumber and sound speed, and primes denote fluctuations. The overbar represents the time average, $\bar{\rho}$ the undisturbed density, $E = -\bar{\rho}c^2\overline{\alpha'\rho'} > 0$ the acoustic energy or wave-energy, $\dot{E}$ its dissipation rate (inverse timescale κk^2), and $\dot{q}_1 = \dot{E}/\bar{\rho}c$ the corresponding dissipation rate of quasimomentum or pseudomomentum $\dot{q}_1$. These are defined positive for positive dissipation.

Before turning to the case of internal gravity waves, we note briefly an example that brings out the theme of this paper is another way and shows clearly why it is worth going to the trouble of thinking in terms of vorticity. In a *non-dissipating* beam of sound in strictly irrotational, homentropic flow, the mean density can be shown to be less than the density in the surrounding fluid, because of the dilatational part of the radiation stress, i.e. the isotropic, δ_{ij} term proportional to the thermodynamic derivative $\partial \log c/\partial \log \rho$ (e.g. Brillouin 1925; Bretherton 1971). More precisely, the time-averaged volume of a material fluid element of given mass can be shown to be greater within the beam than outside it, by an amount that is $O(a^2)$ and proportional to $\partial \log c/\partial \log \rho$. There is an important piece of underwater acoustic technology, the parametric acoustic array, that relies partly on this effect (Westervelt 1963, 1977; McIntyre 1981). It might be thought that the fluid in the beam would therefore be buoyant, and that a vertical mean motion would ensue. But such a mean motion would involve vorticity, and would therefore be impossible, as long as the waves are not dissipating. Then Kelvin's circulation theorem, and (5.2), apply, and the motion remains irrotational. The result can be confirmed by a full analysis using the radiation stress concept (Bretherton 1971, §6; Andrews & McIntyre 1978, §8.4). In the case where the beam is directed horizontally, for instance, what happens is that the buoyancy force, although real enough, is exactly cancelled by a radiation stress divergence due to the refraction, hence concave-upward bending, of the beam in the slight vertical gradient of sound speed caused by gravity.

6. Upgradient PV transport by dissipating internal gravity waves

Figure 4 shows a mean flow induced by internal gravity waves that is a close analogue of the quartz wind. The waves are arriving from below and dissipating at the level shown, in a region of limited extent in y. They are periodic in x, and progressive in the positive x direction, i.e. from left to right. They cause a transverse, upgradient transport of PV along θ-surfaces, in the sense suggested by the large open arrows. The process of creating such a pattern of positive and negative anomalies in the exactly conserved quantity, PV, may be compared to the creation of dipolar patterns in the exactly conserved quantity, electric charge (Obukhov 1962), during electron or positron 'pair production'. The resulting mean flow is very like that generated in the Plumb–McEwan experiment described in §2.

The correlations on the right of (4.9) that give rise to this PV transport are again robust, being insensitive to details of the averaging, for instance whether we average

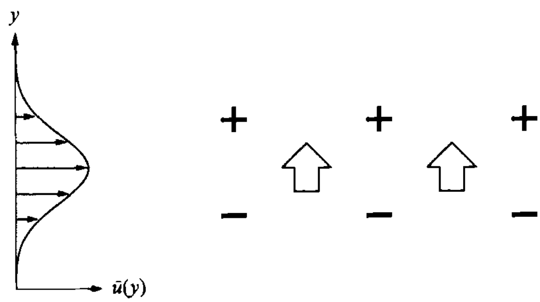

FIGURE 4. Sketch showing the planform of a jet induced by dissipating internal gravity waves, together with the associated PV anomaly pattern (shown by the plus and minus signs) and the upgradient PV transport that produces it (large open arrows).

strictly horizontally, or along undulating θ-surfaces, provided only that additivity over subdivided areas of control surfaces such as the xz plane is preserved. To see how the correlations arise, take the simplest relevant model, namely a plane internal gravity wave with disturbance velocity $\boldsymbol{u}'$ and potential temperature θ' proportional to $\exp[ik(x-ct)+mz]$, with $k>0$, $m<0$ and horizontal phase speed $c>0$. This is the case where phase progresses to the right and downwards, and group velocity (perpendicular to phase velocity and collinear with $\boldsymbol{u}'$) is directed to the right and upwards.

If the waves are dissipating viscously, then the viscous force $\boldsymbol{F}'$ per unit mass is in antiphase with $\boldsymbol{u}'=(u',0,w')$ and is given by

$$\boldsymbol{F}'=-\nu|\boldsymbol{k}|^2\boldsymbol{u}', \tag{6.1}$$

where ν is the kinematic viscosity and $|\boldsymbol{k}|^2$ the squared magnitude of the wavenumber $\boldsymbol{k}=(k,0,m)$. The fluctuation $\nabla\theta'$ in $\nabla\theta$ on the right of (4.9) is

$$\nabla\theta'=\frac{\bar{\theta}N}{g}(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{y}}\cdot\boldsymbol{u}')\,\boldsymbol{k}, \tag{6.2}$$

where $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{y}}$ are unit vectors in the $\boldsymbol{k}$- and y-directions respectively, $N^2=g\bar{\theta}^{-1}|\boldsymbol{G}|$ with g the gravity acceleration and $|\boldsymbol{G}|=\mathrm{d}\bar{\theta}/\mathrm{d}z$, so that N is the buoyancy or Brunt–Väisälä frequency of the basic stable stratification $\bar{\theta}(z)$; $\bar{\theta}$ is taken approximately constant when not differentiated (local Boussinesq approximation). Thus $\boldsymbol{F}'$ and $\nabla\theta'$ are perpendicular and in phase, and the wave-induced contribution on the right of (4.9) is

$$\boldsymbol{J}^{\mathrm{w}}=-\boldsymbol{F}'\times\nabla\theta'=\frac{\nu\bar{\theta}N}{g}|\boldsymbol{k}|^3\,|\boldsymbol{u}'|^2\hat{\boldsymbol{y}}. \tag{6.3}$$

Thus the effective mean force per unit mass given by (4.12) is

$$\bar{\boldsymbol{F}}_{\mathrm{eff}}=\frac{\nu}{N}|\boldsymbol{k}|^3\,\overline{|\boldsymbol{u}'|^2}\hat{\boldsymbol{x}}, \tag{6.4}$$

where $\hat{\boldsymbol{x}}$ is a unit vector in the x direction. This viscous contribution is the only significant contribution in the Plumb–McEwan experiment.

If on the other hand the wave the wave dissipation is purely thermal, as may be an appropriate model in some astrogeophysical applications, then the relevant contribution on the right of (4.9) becomes

$$\boldsymbol{J}^{\mathrm{w}}=-H'\zeta'\hat{\boldsymbol{y}}, \tag{6.5}$$

where ζ' is the vorticity due to the wave motion, defined by

$$\zeta^{a\|} = \nabla \times u' = \zeta' \hat{y}$$

(to leading order in wave amplitude a). H' is the diabatic rate of change of θ', for instance proportional to $-\theta'$ in a Newtonian cooling model or proportional to $\nabla^2\theta' = -|\boldsymbol{k}|^2\theta'$ in a thermal diffusion model. If we write $H' = -\lambda\theta'$ (λ real) and use the facts that the amplitude of ζ' is $|\boldsymbol{k}|$ times the amplitude of $\boldsymbol{u}'$ in magnitude, and that ζ' is in phase with $\theta' = g^{-1}\bar{\theta}N\zeta'/|\boldsymbol{k}|$, we see that the effective mean force per unit mass becomes

$$\bar{F}_{\text{eff}} = \frac{\lambda}{N}|\boldsymbol{k}|\,\overline{|\boldsymbol{u}'|^2}\hat{\boldsymbol{x}}. \tag{6.6}$$

A corollary is that a thermally *excited* wave gives rise to a mean force in the opposite sense, as was pointed out by Lindzen (1973).

It remains to verify that the expressions (6.4) and (6.6) are equal to the horizontal projection of $\dot{\boldsymbol{q}}$, where $\dot{\boldsymbol{q}}$ is the appropriate rate of dissipation of quasimomentum or pseudomomentum $\boldsymbol{q}$. In the plane-wave approximation we may take $\boldsymbol{q} = E\boldsymbol{k}/\bar{\rho}\omega$, where $\omega = kc$ and E is the wave-energy density,

$$E = \tfrac{1}{2}\bar{\rho}\left(\overline{|\boldsymbol{u}'|^2} + \frac{g^2\overline{\theta'^2}}{\bar{\theta}^2N^2}\right) = \bar{\rho}\overline{|\boldsymbol{u}'|^2} = \bar{\rho}\frac{g^2\overline{\theta'^2}}{\bar{\theta}^2N^2}, \tag{6.7}$$

whose mean dissipation rates for the viscous and thermal cases are respectively

$$E = \nu\bar{\rho}|\boldsymbol{k}|^2\overline{|\boldsymbol{u}'|^2}, \tag{6.8a}$$

and

$$\dot{E} = \lambda\bar{\rho}\frac{g^2\overline{\theta'^2}}{\bar{\theta}^2N^2} = \lambda\bar{\rho}\overline{|\boldsymbol{u}'|^2} \tag{6.8b}$$

from (6.7), defined positive as before. Now $\hat{\boldsymbol{x}}\cdot\boldsymbol{k} = k$, and so

$$\hat{\boldsymbol{x}}\cdot\dot{\boldsymbol{q}} = \frac{\dot{E}k}{\bar{\rho}\omega} = \frac{\dot{E}|\boldsymbol{k}|}{\bar{\rho}N}, \tag{6.9}$$

from the dispersion relation $\omega = Nk/|\boldsymbol{k}|$. Together with (6.8$a$, b) this verifies that (6.9) agrees with (6.4) and (6.6).

The net effect of the inhomogeneous turbulence in a *breaking* internal gravity wave must also be to cause upgradient, or gradient-independent, transport of PV along θ-surfaces. We know from laboratory experiments that breaking gravity waves do induce mean flows like that in figure 4 – a well documented example is reported in Delisi & Dunkerton (1989) – and there is every reason to suppose that the same thing happens at arbitrarily high Reynolds numbers. There is a large body of corroborative evidence from observations of momentum fluxes in atmospheric gravity waves, reviewed in Palmer *et al.* (1986, §3). The implication is that irreversible, upgradient PV transport along θ-surfaces must be brought about by contributions to (4.9) from molecular diffusion at the turbulent microscales. This is very different from the assumption, sometimes made in the meteorological and geophysical literature, that PV behaves like a gaseous chemical tracer when turbulent mixing is taking place. The effects of breaking internal gravity waves can be summarized epigrammatically by saying that they transport entropy downgradient across θ-surfaces,† but PV upgradient, or in a gradient-independent sense, along θ-surfaces.

† Albeit often rather weakly, as discussed in McIntyre (1987, 1989b) and references therein; see also the recent laboratory evidence in Delisi & Dunkerton (1989).

7. Breaking Rossby waves and the global atmospheric circulation

Rossby waves are themselves balanced motions, in the sense required for PV inversion. The restoring mechanism giving rise to wave propagation is associated with material conservation of PV, and is brought into play when fluid elements are displaced across a basic isentropic gradient of PV, i.e. a gradient of PV on θ-surfaces (e.g. Hoskins *et al.* 1985, §6*a*). This basic gradient, whose direction will be designated 'northward', gives the problem a rather different character from those considered so far. One aspect is that the irreversible PV transport due to dissipating waves tends to be down the background gradient, i.e. 'southward' along the θ-surfaces.

The most straightforward illustration is the 'breaking' of Rossby waves, in which the PV is simply rearranged advectively in a more or less irreversible way. The PV pattern in figure 2(*c*, *d*) provides a snapshot illustrating this phenomenon in the single-layer model atmosphere, at very large although not wholly unrealistic amplitudes. The basic PV gradient is visible as the pole-to-equator grayscale gradient in figure 2(*d*), with most of it concentrated into a highly contorted band of strong gradients in middle latitudes. By contrast, the opposite extreme, a small-amplitude Rossby wave describable by linear theory, would have PV contours lying nearly east-west along latitude circles, and departing from their basic east-west orientation by gentle sideways undulations only. Breaking Rossby waves may be compared and contrasted with breaking gravity waves. Whereas the latter tend (i) to generate three-dimensional turbulence, (ii) to deform isentropic surfaces irreversibly, and (iii) to rearrange entropy downgradient in the vertical, breaking Rossby waves tend (i) to generate layerwise-two-dimensional or so-called 'geostrophic' turbulence, (ii) to deform PV contours irreversibly, and (iii) to rearrange PV downgradient in the 'north-south' direction, along isentropic surfaces.

In the real atmosphere, as in models like that of figure 2, Rossby wave breaking appears to be a very common, and often important, process. Other processes such as radiative heat transfer may interact significantly with the advective rearrangement (e.g. Butchart & Remsberg 1986; Butchart 1987; HM §3, O'Neill & Pope 1987; Salby *et al.* 1989), but often do not drastically change its nature, the net effect being some amount of wave dissipation that is partly due to the irreversible advective rearrangement of PV down to fine spatial scales – the kind of turbulent random-straining process discussed by Batchelor (1952, 1959), Cocke (1969), Kraichnan (1974, 1975) and others – and partly due to the radiative heat transfer. The pattern of mean forces due to breaking or thermally dissipating gravity waves, for instance mountain waves, may also interact with the whole process. The conservation and 'impermeability' properties discussed in HM and in §4 imply, once again, that the net effect of all this on the PV field can be described in terms of a transport of PV oriented exactly along isentropic surfaces.

The simplest theoretical paradigm for the Rossby wave breaking process is that provided by nonlinear Rossby-wave critical layer theory, in which most of the wave breaking is confined to a narrow zone or 'critical layer' oriented east-west and surrounding the 'critical line', or location at which the intrinsic wave frequency vanishes because of shear in a pre-existing east-west mean flow. In the cases most thoroughly studied, the fluid system is taken to be simple two-dimensional inviscid vortex dynamics, i.e. the dynamical system associated with (3.1) but at limitingly small Froude number so that h becomes constant and Q and ζ^a become the same thing. Although the resulting model is not closely realistic for the atmosphere or oceans, the problem keeps the same basic qualitative structure while becoming relatively tractable mathematically, in part because it has a very simple PV inversion operator, the inverse Laplacian (giving a stream function for the velocity field when applied to Q), and in part because of the spanwise scale separation between the narrow north-south scale of the main breaking region and the much broader scale of its (relatively undular) surroundings.

The outcome is that one can describe the entire nonlinear evolution in a very clear-cut way, in terms of the interaction between the two regions, using the method of matched asymptotic expansions. The first such analysis, in a special case giving the simplest self-consistent model example of the Rossby wave breaking process, was given in the pioneering work of Stewartson (1978) and Warn & Warn (1978), hereafter 'SWW'. The importance of the problem is its status as one of the few self-consistent thought experiments within which one can explore some aspects of the kind of highly inhomogeneous 'wave-turbulence jigsaw puzzle', involving neighbouring undular and wave breaking regions, that seems typical of large-scale oceanic and atmospheric fluid motions. It also serves, incidentally, as a particularly clear illustration of the way in which downgradient PV transport can, indeed, take place in geophysical fluid systems, despite the seeming paradoxes that arise in thought experiments where PV mixing is considered without taking into account the possibility of an associated radiation stress, or wave-induced momentum transport from outside the mixing region (e.g. Stewart & Thomson 1977). A detailed discussion of the nature of the interaction between the regions, both in mathematical and in physical terms, can be found in the paper by Killworth & McIntyre (1985), with emphasis on the way in which the wave breaking region contrives to absorb just the amount of momentum required by the downgradient PV rearrangement.

Even in the idealized context of the critical layer theory, the details of the wave breaking process can be very complicated, in cases less special than that of SWW. Figure 5(*a*) shows the Q field in an example where small-scale instabilities have been allowed to grow and play their part in the evolution, from a recent and comprehensive study of the problem by Haynes (1989). Substantial irreversible rearrangement of the Q field is taking place. Figure 5(*b*) shows the Eulerian-mean Q profile together with the original, undisturbed Q profile (shown dashed), and figure 5(*c*) the resulting change in the Eulerian-mean flow. Qualitatively similar mean-flow changes in the same sense (not shown here) are obtained both in the SWW problem and in more realistic, large-amplitude examples like that of figure 2. In an Eulerian average taken around latitude circles, in the case of figure 2, there is a substantial mean westward flow in the tropics, where the PV has been rearranged most drastically by the wave breaking. The numerical experiments show that this mean flow was not present initially and that it is generated in fundamentally the same way as in the SWW problem and in the critical-layer problem of figure 5, the only difference being that, because the wave amplitude is so much larger, the irreversible PV transport and mean flow change take place over a much broader region. Another such large-amplitude example is documented in greater detail in a forthcoming paper by Juckes *et al.* (1990; see also Juckes & McIntyre 1987).

Figure 5(*c*) and the other examples mentioned remind us, among other things, of the well known *one-signedness* of the effective mean force and irreversible mean-flow tendency that result from Rossby-wave breaking. Inasmuch as the PV transport tends to be downgradient and therefore southward, the effective mean force tends to be westward (recall (4.12)). This fact is central to understanding many aspects of the

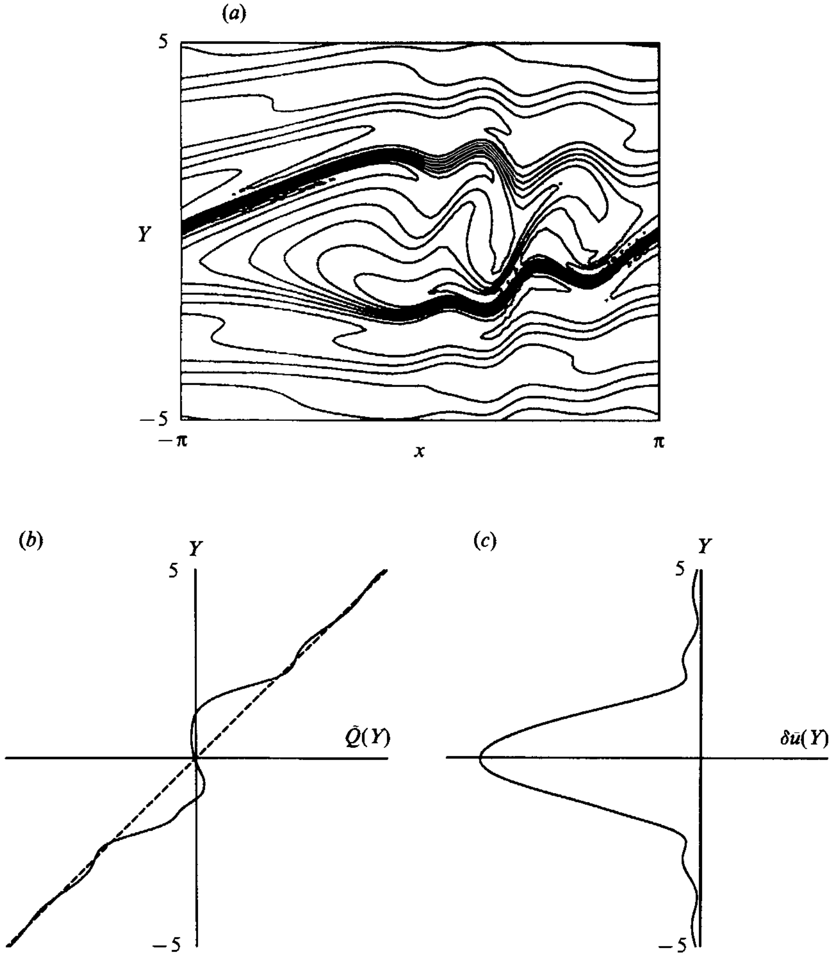

FIGURE 5. (*a*) Contours of Q or ζ^a in the narrow wave breaking or critical layer region in an inviscid, nonlinear Rossby-wave critical layer simulation like those described in Haynes (1989), except at higher numerical resolution. A combination of analytical and numerical techniques is used (both here and in the reference cited) in order to achieve this resolution. The width of the critical layer region is exaggerated for clarity. The contours in the central region of moderate values of Y, the stretched y or northward coordinate, are being irreversibly deformed to an extent that leads to a substantial irreversible rearrangement of Q and hence a substantial change in the eastward mean flow $\bar{u}(Y)$. (*b*) Eulerian-mean initial (dashed) and present (solid) $\bar{Q}(Y)$ profiles, differing by $\delta\bar{Q}(Y)$ say. (*c*) Eulerian-mean flow change $\delta\bar{u}(Y)$ that results from the rearrangement. In this simple model the inversion operator that gives $\delta\bar{u}$ in terms of $\delta\bar{Q}$ is simply minus the y-integral of $\delta\bar{Q}$.

global-scale atmospheric circulation, including not only the ozone and chlorofluorocarbon problems but also, for instance, the distribution of prevailing surface winds, and the related fact, once regarded as one of the major enigmas in the atmospheric sciences (e.g. Lorenz 1967, pp. 85, 150), that angular momentum is observed to be transported against its own mean gradient in the subtropical stratosphere and upper troposphere (e.g. Gill 1982, §§13.9, 13.10; Hoskins *et al.* 1985, §§6*d*, 9).

It is beyond the scope of this paper to discuss in detail the role of the various wave-induced mean forces in the three-dimensional global circulation and the resulting transport of chemical constituents – for recent discussions of different aspects see for instance WMO (1985), Fels (1985), HM, Andrews *et al.* (1987), Dunkerton (1989), McIntyre (1989*a*, *b*), and Haynes *et al.* (1990) – but one can say briefly that their importance for the circulation stems from the fact that the Earth is a rapidly rotating planet. Rapid rotation means here that the distribution of azimuthally averaged angular momentum in the atmosphere is dominated by the Earth's rotation, in extratropical latitudes. Systematic wave-induced or other mean forces give rise to systematic mean motions across the constant angular momentum surfaces. The mean vertical motions needed for mass continuity are important, in turn, in carrying chemical constituents between the lower atmosphere and the photochemically most active regions above about 25 km, and hence, for instance, in determining the rate of destruction of man-made chlorofluorocarbons introduced into the lower atmosphere.

The adiabatic heating and cooling associated with the same vertical motions also hold temperatures T away from their radiatively determined values T_{rad}, a notion that makes qualitative sense (even though T_{rad} is itself somewhat affected by the transport of chemical constituents) because of the substantial magnitude of the typical temperature anomaly $(T - T_{rad})$ and because of the generally relaxational character of radiative heating and cooling, which broadly speaking has a tendency to reduce $|T - T_{rad}|$. This set of ideas goes back to the pioneering work of Dickinson (1969*a*) and appears to be especially pertinent in the stratosphere and mesosphere, indeed throughout the 'middle atmosphere' up to 100 km or so (e.g. WMO 1985, chapter 6). Current thinking assigns a dominant role to upward propagating Rossby waves and related types of large-scale disturbances; internal gravity waves are believed to be the next most important except at very high altitudes, above the stratopause (> 50 km), where they tend to become the dominant type (e.g. Fritts 1984, 1987; Andrews *et al.* 1987, and references therein). Both kinds of waves are envisaged as being generated largely by nonlinear processes in the troposphere. Many of the wave sources involve distinctively tropospheric effects such as cumulonimbus convection, flow over topography, land-sea contrasts, and large-scale cyclogenesis, itself related to Rossby-wave dynamics and also dependent on the strong surface potential-temperature and humidity gradients typical of the lower troposphere (e.g. Gill 1982; Hoskins *et al.* 1985, §6).

8. Thermally dissipating Rossby waves and the Antarctic ozone hole

At an opposite extreme to the idealized case of figure 5, in which the Rossby waves are breaking without thermally dissipating, is the converse idealization in which they are thermally dissipating without breaking. This classical problem, first studied by Dickinson (1969*b*), has its own relevance to the dynamics of the global atmospheric circulation, because of the relaxational character of the radiative heat transfer already mentioned. It also bears directly on current questions about the effectiveness of the Rossby-wave restoring mechanism in inhibiting chemical transport across bands of strong PV gradients on isentropic surfaces, where the restoring mechanism may be locally strong enough to suppress Rossby-wave breaking. Such an undular 'PV barrier' against chemical transport on isentropic surfaces is believed to occur for instance at the edge of the wintertime polar-night vortex, and to be crucial to the formation of the Antarctic ozone hole, whose chemistry appears to require a large degree of isolation of the air within the vortex from its surroundings. An urgent and currently controversial question is how complete or otherwise this isolation might be (e.g. Hartmann *et al.* 1989; Murphy *et al.* 1989; Proffitt *et al.* 1989; Tuck 1989), and in particular how the interplay between dynamics and radiative heat transfer might bring about systematic mean mass and chemical transports across the edge of the vortex. These questions will be fully discussed in a forthcoming paper by Haynes & Norton (1990); but since they are also part of the general theme of this essay, a few of the relevant considerations are briefly sketched here in the context of the idealization of purely thermal dissipation.

The most direct way of describing the thermally dissipating, stably stratified, three-dimensional flows of interest, at the same time as seeing their connection with single-layer models like those of figures 2 and 5, is to formulate the problem in isentropic coordinates, assuming hydrostatic balance. We use the potential temperature θ as the vertical 'coordinate'. The general PV conservation equation then has the same appearance as (4.8), namely

$$\frac{\partial(\sigma Q)}{\partial t}+\nabla\cdot\boldsymbol{J}=0, \tag{8.1}$$

but the flux $\boldsymbol{J}$ takes the simple form

$$\boldsymbol{J}=(u,v,0)\,\sigma Q+\boldsymbol{J}_H+\boldsymbol{J}_F \tag{8.2}$$

(HM equation (2.5)), the two contributions $\boldsymbol{J}_H$ and $\boldsymbol{J}_F$ being given to within the hydrostatic approximation by, respectively, $\boldsymbol{J}_H=(H\,\partial v/\partial\theta,\ -H\,\partial u/\partial\theta,0)$, the contribution from the local rate of diabatic heating $H=\mathrm{D}\theta/\mathrm{D}t$, and $\boldsymbol{J}_F=(-G,F,0)$, the contribution from an arbitrary body force $\boldsymbol{F}$ with horizontal components F,G. The zeros express the impermeability theorem noted in HM and in §4. Q is the Rossby-Ertel PV for the stratified fluid, and σ the apparent mass density in '$xy\theta$-space' such that $\sigma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta$ is the mass element, the mass-conservation equation in isentropic coordinates therefore being

$$\frac{\partial\sigma}{\partial t}+\nabla\cdot(\sigma u,\sigma v,\sigma H)=0. \tag{8.3}$$

The symbol ∇ now stands for $(\partial/\partial x,\partial/\partial y,\partial/\partial\theta)$. We have $\sigma Q\doteq\zeta^{\mathrm{a}}$ by definition, where ζ^{a} is defined as $\partial v/\partial x-\partial u/\partial y$ plus the vertical component of $\boldsymbol{\Omega}$. The derivatives $\partial/\partial t$, $\partial/\partial x,\partial/\partial y$ are now always taken at constant θ. The components u, v and F, G are the true horizontal components, and not components along θ surfaces (e.g. Andrews *et al.* 1987). Following HM and using (8.3) we may rewrite (8.1) as

$$\frac{\mathrm{D}_\theta Q}{\mathrm{D}t}=\frac{Q}{\sigma}\frac{\partial(\sigma H)}{\partial\theta}-\frac{\nabla\cdot\boldsymbol{J}_H+\nabla\cdot\boldsymbol{J}_F}{\sigma}=\varDelta,\ \text{say}, \tag{8.4}$$

where $\mathrm{D}_\theta/\mathrm{D}t$ is the quasi-material rate of change following an 'isentropic trajectory', i.e. moving with horizontal velocity (u,v) but staying on one isentropic or constant-θ surface:

$$\frac{\mathrm{D}_\theta}{\mathrm{D}t}=\frac{\partial}{\partial t}+(u,v,0)\cdot\nabla=\frac{\partial}{\partial t}+\boldsymbol{u}_{\mathrm{h}}\cdot\nabla,\quad \text{say}.$$

Since the waves are now assumed not to be breaking, it is reasonable for the purposes of qualitative understanding to use linearized wave theory, as we did in §§5, 6. Linearizing (8.4) about an x-independent mean state $Q=\bar{Q}$, $\sigma=\bar{\sigma}$, $\boldsymbol{u}_{\mathrm{h}}=(\bar{u},0,0)$, we have

$$\bar{\mathrm{D}}Q'+v'\frac{\partial\bar{Q}}{\partial y}=\varDelta', \tag{8.5}$$

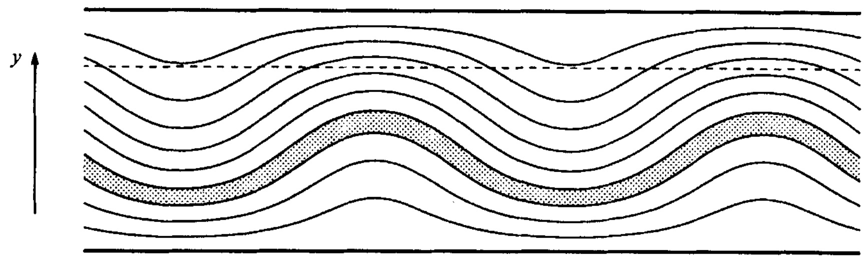

FIGURE 6. Isentropic quasi-material contours in a simple, thermally dissipating Rossby wave that is not breaking, i.e. the contours are simply undulating and not deforming irreversibly. The contours shown actually represent a finite-amplitude solution describing a steady-state, quasi-geostrophic, weakly dissipating Rossby wave in a beta-channel with constant $\bar{u}$ and constant $\partial\bar{Q}/\partial y$; equations (8.5)–(8.8) consider only those aspects described by linearized theory, but equations (8.9) ff. are relevant at finite amplitude. The dashed line is a line of constant latitude y, and the shading picks out a typical quasi-material tube, as used in the analysis of equation (8.11).

where primes denote disturbances about the mean state, and the basic flow material derivative is defined as

$$\bar{\mathrm{D}} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}.$$

As usual in this kind of theory we have set $\bar{v}$ to zero since it is formally consistent to regard $\bar{v}$ as part of the $O(a^2)$ mean response and therefore negligible in (8.5), where a is the wave amplitude as before, assumed small in an appropriate sense. Define also the northward quasi-material displacement η' such that

$$\bar{\mathrm{D}}\eta' = v'. \tag{8.6}$$

Then a few lines of manipulation give two alternative expressions for the northward eddy transport of PV divided by $\bar{\sigma}$ (cf. (8.8) below), namely

$$\overline{v'Q'} = -\overline{\eta'\Delta'} + \frac{\partial}{\partial t}\left(\overline{\eta'Q'} + \tfrac{1}{2}\overline{\eta'^2}\frac{\partial\bar{Q}}{\partial y}\right), \tag{8.7a}$$

$$= \left[\overline{Q'\Delta'} - \frac{\partial}{\partial t}(\tfrac{1}{2}\overline{Q'^2})\right]\bigg/\left(\frac{\partial\bar{Q}}{\partial y}\right). \tag{8.7b}$$

Here the overbars denote average over x at fixed y and θ; we call this a quasi-Eulerian average. The first expression comes from multiplying (8.5) by η' and using (8.6). The second expression is the isentropic coordinate version of a result found by Holton & Dunkerton (1978); it comes from multiplying (8.5) by Q'.

Figure 6 shows some of the quasi-material contours defined by the displacement field η', or rather its finite-amplitude counterpart, for an example of the kind of simple, purely undular, non-breaking Rossby wave we have in mind. For weak dissipation these quasi-material contours will almost coincide with Q contours in the θ-surface, but not exactly, because of the dissipation term Δ' on the right of (8.5). Note further that, in all the standard theories at least, the linearized inversion operator that complements equation (8.5) and closes the problem is taken to be independent of the dissipation term Δ' to sufficient accuracy, as well as, of course, being independent of t. So the right hand side of (8.5) is the only place where the thermal, or any other, dissipation appears in the linearized wave theory. It follows that if we have a steady wave solution $Q' = \tilde{Q}(\boldsymbol{x})$ to the non-dissipative problem $\Delta' = 0$, then, for instance, the textbook dissipative problem that has $\Delta' = -\mu Q'$ with $\mu = \text{constant}$ is solved by $Q' = \tilde{Q}\,\mathrm{e}^{-\mu t}$.

More generally, a non-breaking Rossby wave that is dissipating, weakly enough to retain the character of a propagating wave, is a wave in which PV contours and quasi-material contours are undular as in figure 6 and in which the term Δ' on the right of (8.5) is at least roughly in phase with η', and in antiphase with Q', giving a negative contribution to $\overline{v'Q'}$ from the terms involving Δ' in (8.7). Thus it follows from either of (8.7a, b) that the $O(a^2)$ advective PV transport $\bar{\sigma}\overline{v'Q'}$ tends to be negative or downgradient in an approximately steady, dissipating Rossby wave. Standard scaling considerations such as those given in HM show moreover that, in the usual quasi-geostrophic scaling regime, the $\boldsymbol{J}_H$ term in (8.2) is relatively negligible; and we ignore $\boldsymbol{J}_F$ once we assume that the dissipation is purely thermal. Thus, quite unlike a thermally dissipating gravity wave, a thermally dissipating Rossby wave transports PV downgradient and, again quite unlike a thermally dissipating gravity wave it does so in a predominantly advective manner.

Note that the $\partial/\partial t$ terms in (8.7) show, in addition, that a non-dissipating Rossby wave also transports PV downgradient when its amplitude is growing, although this transport would subsequently reverse if the wave propagated away without breaking (cf. the case study by Palmer & Hsu 1983). Equally, these terms are relevant to the incipient stages of the wave-breaking process discussed in §7, and show the *beginning* of the downgradient PV transport process in that problem. Various scenarios suggest themselves in which wave amplitude fluctuates but in which, if no wavebreaking occurs, the $\partial/\partial t$ terms in (8.7) tend to average out as time goes on, leaving the systematic downgradient $\overline{v'Q'}$ due to the thermal dissipation Δ' as the dominant effect in the long time average. The corresponding effective mean force is obtained by applying the arguments of §4 to (8.2) ff., to give $\bar{\boldsymbol{F}}_{\text{eff}} = (\bar{F}_{\text{eff}}, 0, 0)$, where

$$\bar{F}_{\text{eff}} = \overline{(\sigma v)'Q'} \approx \bar{\sigma}\overline{v'Q'}, \tag{8.8}$$

whose tendency to be negative recalls the discussion in the second half of §7.

These results on the behaviour of the isentropic eddy transport $\overline{v'Q'}$ are of a standard kind but are derived here to show how the case of thermally dissipating Rossby waves fits into our general characterization. It does so provided we interpret 'mean' in a suitable Eulerian or quasi-Eulerian sense, as above. The importance of this latter point, which makes the Rossby-wave problem very different from the problems of §§5, 6, can be brought out more clearly by a brief consideration of the finite-amplitude aspects, continuing to neglect $\boldsymbol{J}_H$ and $\boldsymbol{J}_F$.

The simplest relevant thought experiment envisages a steady wave source somewhere below the isentropic layer of interest, and a steady dissipating wave field. We assume that the wave-mean system has settled down as a whole to an exactly steady state in which not only is the wave amplitude steady, but also the mean circulation. This makes sense because of the relaxational character of the radiative heat transfer. As pointed out in HM (*q.v.*, equation (3.5)), the dominant balance is then

$$\overline{\sigma v}\,\bar{Q} = -\overline{(\sigma v)'Q'} \approx -\bar{\sigma}\overline{v'Q'}. \tag{8.9}$$

This follows from integrating (8.1) over an area lying between the northern boundary and a given latitude such as the dashed line in figure 6, continuing to neglect $\boldsymbol{J}_H$ and $\boldsymbol{J}_F$, setting $\partial/\partial t = 0$ or taking a time average, and using the divergence theorem. Equation (8.9) is an alternative way of expressing what was referred to in §7 as the wave-driven mean circulation across angular momentum surfaces, or at least the part of it driven by Rossby waves in a steady or statistically steady state. The tendency already noted for the right hand side of (8.8) to be negative, i.e. for the right hand side of (8.9) to be positive, at least in a long time average, implies a tendency for the mean mass circulation $\overline{\sigma v}$ to be persistently poleward. This agrees with the observational evidence for the lower stratosphere, and the middle stratosphere in winter, where Rossby-type disturbances are thought to be dominant (e.g. Townsend & Johnson 1985; WMO 1985), and is consistent with the standard leading-order theoretical description based on quasi-geostrophic scaling (WMO 1985; Andrews *et al.* 1987).

Does this also bear on the question about mean mass transport across 'PV barriers' such as the edge of the Antarctic ozone hole in the lower stratosphere, when PV contours are undulated and subject to thermal dissipation? The answer is no, because this latter question is a Lagrangian and not an Eulerian question. The distinction is now critical, essentially because the sideways displacements of PV and quasi-material contours are an essential part of the Rossby-wave problem.

To make this clearer we continue to neglect $\boldsymbol{J}_H$ and $\boldsymbol{J}_F$, but now integrate equation (8.1) over an area lying between the northern boundary and a given PV contour, Γ say, with value Q_Γ. Again using the two-dimensional divergence theorem we now see that

$$Q_\Gamma \int_\Gamma \sigma \boldsymbol{u}_\mathrm{h} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}s = 0 \tag{8.10}$$

in the steady state, where $\boldsymbol{u}_\mathrm{h}$ again denotes $(u, v, 0)$ and $\hat{\boldsymbol{n}}$ is a unit horizontal vector normal to the PV contour. This result was first pointed out to us by P. H. Haynes (personal communication). It implies that in the steady state and with the customary neglect of $\boldsymbol{J}_H$ and $\boldsymbol{J}_F$, the net mass transport across any PV contour must vanish. Another way to say this is that quasi-material contours like those in figure 6 do not systematically drift northwards or southwards. That is, they behave just like the steady-state PV contours apart from the fact that their undulations are phase-shifted slightly eastwards.

The fact that there is no net diabatic mass transport across a PV barrier, according to this model, contrasts with (8.9) and illustrates the well known way in which Eulerian-mean and Lagrangian-mean descriptions of the same thing can look strikingly different. For the ozone-hole problem this focuses attention on the factors neglected in the present argument, such as $\boldsymbol{J}_H$, $\boldsymbol{J}_F$, and wave breaking; the full implications are discussed in Haynes and Norton (1990).

The way in which (8.10) fits in with (8.9), and with the PV conservation and impermeability properties, can also be viewed as follows. First, note that if (8.1) is now integrated over a quasi-material region such as the shaded band in figure 6, or any other quasi-material region, i.e. a region on the θ-surface that moves with the velocity field $\boldsymbol{u}_\mathrm{h}$, then

$$\iint \sigma Q\, \mathrm{d}x\, \mathrm{d}y = \text{constant}, \tag{8.11}$$

since under the assumed conditions (8.1) says that the PV transport is purely advective. If, as in HM, we think of the PV as the mixing ratio, or amount per unit mass, of a peculiar chemical 'substance', then (8.11) can be pictured as saying that the total amount of such 'PV-substance' contained in any quasi-material volume lying between two isentropic surfaces $\theta = \theta_1, \theta_2$ cannot change (HM equation (2.15)) – a consequence of the 'impermeability theorem' and valid, incidentally, whether or not the motion is steady. The result applies not only to the quasi-material tube corresponding to the shading in figure 6, but also to any quasi-material element being moved along it by the velocity field $\boldsymbol{u}_{\mathrm{h}}$.

It is the fact that the θ-surfaces are permeable to mass, even if not to PV, that accounts for the fluctuations in Q despite (8.11). The wave dissipation process implies alternate dilution and concentration (HM §2) of the 'PV-substance' contained in a small quasi-material element as it moves along a quasi-material tube. The phase of the resulting fluctuations in Q has already been noted in the discussion leading to (8.8). The negative $\overline{v'Q'}$ correlation implies that dilution, or mass inflow, rates peak somewhere near the southernmost excursions of the tubes, and that concentration, or mass outflow, rates peak somewhere near the northernmost excursions. This means that, by mass conservation, the northward-flowing parts of the quasi-material tube carry more mass than the southward-flowing parts. Hence there is a net northward mass flow across any latitude $y=$ constant, such as the dashed line in figure 6. It is this, and only this, that gives rise to the positive value of $\overline{\sigma v}$ implied by (8.7) and (8.9) in the steady state.

The way in which the mass flow closes vertically is of interest in connection with the global atmospheric and photochemical processes touched on in §7. In the present model it is governed by the result of multiplying (8.4) (with J_H and J_F neglected) by σ/Q and integrating with respect to θ, namely

$$\rho w_{\mathrm{diab}}|_{\theta_0} = \sigma H|_{\theta_0} = -\int_{\theta_0}^{\infty} \frac{\mathrm{D}_\theta \ln Q}{\mathrm{D}t}\,\sigma\,\mathrm{d}\theta, \tag{8.12}$$

where w_{diab} is the diabatic vertical velocity, equal by definition to $\sigma H/\rho$. HM argue that this integral is convergent, in practice requiring integration over only a few density scale heights (see also Haynes *et al.* 1990). This relates the diabatic mass flow across any given isentropic surface $\theta=\theta_0$ to the concentration and dilution rates of tubes lying *above* that surface, those lying below it being irrelevant. Like (8.11), (8.12) is true even if the motion is unsteady, and even if wave breaking is occurring. In our special case of steady waves it says that in the region north of the dashed line of latitude in figure 6, for example, there is a predominance of quasi-material tubes extending northward and expelling mass, resulting in positive $\mathrm{D}_\theta Q/\mathrm{D}t$. This contributes negatively to the right-hand side of (8.12), and therefore contributes to diabatic descent across isentropic surfaces at lower levels, for instance near the tropopause. This result is of interest in view of an assumption sometimes made that the descent is controlled more locally, namely by events at, rather than above, the tropopause (e.g. WMO 1985, chapter 5).

9. The Craik–Leibovich instability

There is an interesting phenomenon, the Craik–Leibovich instability (e.g. Craik 1977, 1982*b*, 1985; Leibovich 1980, 1983), that does not fit neatly into any of the categories discussed in §2. It is probably best to regard it as in a category on its own. It is a likely cause, in many instances, of the longitudinal vortices that are often found in shear flows subject to wavy disturbances of almost any kind. A well known example comprises the 'Langmuir vortices' observed in wind-blown oceans and lakes. The theory suggests that the instability mechanism is a robust mechanism, insensitive to the detailed shape of the shear flow profile. It is governed (in the most-studied asymptotic limit) by equations resembling the equations of thermal convection. The essential conditions for it to occur are the presence of a wavy disturbance having a sheared Stokes drift, together with pre-existing vorticity giving an Eulerian-mean shear in the same sense as the Stokes drift.

The mean flow evolution is 'balanced' in the sense of our earlier discussion, being completely determined by the (advective) transport of vorticity. However, it cannot be called a classical wave-driven mean flow, because there is no essential dependence upon wave breaking nor upon any other dissipative mechanism and, unlike the examples in the past four sections, the vorticity transport cannot in the same way be said to be directly wave-induced. The transport is entirely accounted for by advection by the mean flow, provided that we define this to be the Lagrangian-mean flow, thus including the Stokes drift of the waves. The waves, through their Stokes drift, have a catalytic effect rather than a direct driving effect. In consequence of this, the magnitude of the mean flow change is not uniformly bounded by $O(a^2)$, despite the absence both of wave dissipation and of wavelike mean-flow dynamics.

It appears likely that the magnitude of the mean flow change is limited, instead, by the magnitude of the pre-existing mean vorticity in the initial state. However, the theory has not yet been extended to all the relevant cases in which the initial vorticity is arbitrarily strong, although some progress has been made (Craik 1982*b*). The impediment to progress is the difficulty of calculating the reaction of strong mean-flow changes back upon the waves, which might conceivably distort them enough to kill off their catalytic action at some stage in the evolution. For this and other reasons the ultimate fate of an inviscid flow subject to the instability is still an open question, as is whether, for example, the evolution is irreversible, as it is in the inviscid example of §7, or whether, for example, it is recurrent in the Fermi–Pasta–Ulam sense.

10. Concluding remarks

We have argued that the general nature of the classical, dissipative type of wave-induced mean motion can be understood in a unified way, by viewing all the phenomena in terms of the wave-induced upgradient or downgradient transport of vorticity or PV. The basic examples described in §§5–8 illustrate the transport effects in a manner accessible either to straightforward calculation by analytical methods, or to credible numerical simulation. The suggested picture not only shows the various ways in which wave dissipation, in the general sense including wave breaking, is crucial to these wave-induced mean effects, but also brings more clearly into focus the ultimate limitations of applicability, in its most general form, of the whole idea of classical mean streaming. This idea, and the ideas and intuitions associated with it, apply to the extent that, and to the accuracy with which, the vorticity or PV invertibility principle applies (§3).

The picture thus arrived at complements previous descriptions of wave-mean interaction in terms of radiation stress, pseudomomentum, circulation theorems, and non-acceleration theorems, and gives a succinct yet general way of saying why certain contributions to the radiation stress are significant for mean streaming, and others not. Radiation stress remains useful of course as a natural way of describing the transport of momentum or angular momentum *between* sites of wave generation and wave dissipation, especially if those sites are well separated spatially, as in the example of figure 1 (*b*) and in the ocean beach longshore-current problem (where the generation and dissipation sites may be separated by thousands of kilometres), and in some of the problems that are significant for understanding the general circulation of the atmosphere (§§6–8).

Regarding current atmospheric research, a major challenge at present is to refine our picture of the circulation to include, interdistinguish, and eventually quantify, the more subtle three-dimensional effects, and the mean circulations and chemical transports associated with prominent observed features such as the steep isentropic gradients of PV at the edges of the wintertime stratospheric polar vortices and elsewhere, for instance in the subtropics. The PV conservation and isentropic impermeability theorems appear to provide some important simplifications in our thinking about such problems (§8; Haynes & Norton 1990). It is not known whether, in addition, the three-dimensional effects will turn out to include significant non-dissipative interactions (§2) between the internal gravity and Rossby waves, or between other wave types, as well as the dissipative interactions on which we have concentrated here. Whatever eventuates, it seems safe to assume that descriptions centred around the way in which the general PV conservation equations (4.8), (8.1) are satisfied will be one of the keys to a better understanding of the complex interplay of dynamics, chemistry and radiation controlling the state of our planet's atmosphere.

It is a unique privilege to have been given the opportunity to contribute to this very special and important Festschrift Volume of JFM. It is impossible to say in a few words how much we ourselves, and the subject of fluid dynamics, owe to GKB. Our debt is great and our tribute inadequate. We thank also D. G. Andrews, E. F. Danielsen, P. H. Haynes, B. J. Hoskins, I. N. James, M. N. Juckes, M. J. Lighthill, M. S. Longuet-Higgins, J. Mestel, T. N. Palmer, T. G. Shepherd, G. J. Shutts, A. J. Thorpe and J. Tribbia for stimulating conversations and correspondence about different aspects of the fluid dynamics involved here. MEM would also like to record that his early interest in problems of this kind was stimulated, in different ways, by F. P. Bretherton, D. O. Gough, and E. A. Spiegel. The 'kitchen-sink' demonstrations described in figure 1 are variants of one of M. S. Longuet-Higgins' beautiful wave-tank demonstrations. The relation (8.10) was first pointed out to us by P. H. Haynes. We are grateful to B. J. Hoskins, I. N. James and M. N. Juckes for generous help with the numerical models, which were based on those developed in the Meteorology Department at the University of Reading. S. P. Cooper gave expert help with the task of making high-resolution model output intelligible to the human eye, and in many other ways. A first draft of some of this material formed part of an essay that shared the 1981 Adams Prize in the University of Cambridge. The ideas were further developed with support from the Japan Society for the Promotion of Science, the Commonwealth Scholarship Commission, the Cambridge Commonwealth Trust, the Natural Environment Research Council through the UK Universities' Global Atmospheric Modelling Project (grant GR3/6516) and through grants GR3/5572 and GST/02/446 (British Antarctic Survey), the US Office of Naval Research, and the UK Department of the Environment in connection with the recent Airborne Arctic Stratospheric Expedition, involvement with which has been an important stimulus to our recent thinking about the implications for understanding the Antarctic ozone hole and related phenomena.

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